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COMPLEX MODULI OF VISCOELASTIC COMPOSITES—II. FIBER REINFORCED MATERIALS†

ZVI HASHIN‡

Towne School, University of Pennsylvania, Philadelphia, Pennsylvania

Abstract—A correspondence principle which was established in a previous paper, is applied to derive expressions for effective complex moduli of fiber reinforced materials on the basis of expressions for effective elastic moduli of fiber reinforced materials.

1. INTRODUCTION

IN A preceding paper [1] there has been developed a general theory of complex moduli of viscoelastic composites based on the assumption that within some frequency range macroscopic dynamic behavior may be approximated by classical continuum dynamics.

There has been established a correspondence principle whereby expressions for effective elastic moduli of a composite can be transformed into effective moduli of a viscoelastic composite, simply by replacement of phase elastic moduli by phase complex moduli. The general theory has been developed for general macroscopic anisotropy and it is therefore directly applicable to fiber reinforced materials.

The present work complements a previous investigation of static viscoelastic properties of fiber reinforced materials [2].

2. RÉSUMÉ OF ELASTIC BEHAVIOR OF UNIDIRECTIONALLY FIBER REINFORCED MATERIALS

If the fibers are all parallel and randomly dispersed in the plane of their cross sections the FRM (abbreviation for Fiber Reinforced Material to be used from now on) is macroscopically transversely isotropic. Its elastic stress strain law, in terms of average strains and stresses, is then [3]

$$
\begin{aligned}
\bar{\sigma}_{11} &= C^*_{11}\bar{\varepsilon}_{11}+C^*_{12}\bar{\varepsilon}_{22}+C^*_{12}\bar{\varepsilon}_{33} && \text{(a)}\\
\bar{\sigma}_{22} &= C^*_{12}\bar{\varepsilon}_{11}+C^*_{22}\bar{\varepsilon}_{22}+C^*_{23}\bar{\varepsilon}_{33} && \text{(b)}\\
\bar{\sigma}_{33} &= C^*_{12}\bar{\varepsilon}_{11}+C^*_{23}\bar{\varepsilon}_{22}+C^*_{22}\bar{\varepsilon}_{33} && \text{(c)}\\
\bar{\sigma}_{12} &= 2C^*_{44}\bar{\varepsilon}_{12} && \text{(d)}\\
\bar{\sigma}_{23} &= (C^*_{22}-C^*_{23})\bar{\varepsilon}_{23} && \text{(e)}\\
\bar{\sigma}_{31} &= 2C^*_{44}\bar{\varepsilon}_{31}. && \text{(f)}
\end{aligned}
\qquad (2.1)
$$

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‡ Consultant to the Franklin Institute Research Laboratories, Phila., Pa. On leave at Dept. of Materials Engng. Technion—Israel Institute of Technology, Haifa, Israel.

Here the stress–strain law is in reference to a cartesian system of axes with x_1 in fiber direction.

The stress–strain law (2.1) involves five independent effective moduli. If the material is square symmetric, such as a square array of identical circular fibers, then (2.1e) has to be replaced by

$$\bar{\sigma}_{23} = 2C^*_{55}\bar{\varepsilon}_{23} \tag{2.2}$$

where C^*_{55} is an independent modulus. The rest of (2.1) remains unchanged. It is seen that the square symmetric material has six independent moduli.

Physically important effective elastic moduli are now listed in terms of C^*_{ij} used in (2.1)

$$k^* = \tfrac{1}{2}(C^*_{22}+C^*_{23}) \tag{2.3}$$

$$G^*_t = \tfrac{1}{2}(C^*_{22}-C^*_{23}) \tag{2.4}$$

$$G^*_a = C^*_{44} \tag{2.5}$$

$$E^*_a = C^*_{11} - \frac{2C^{*2}_{12}}{C^*_{22}+C^*_{23}} \tag{2.6}$$

$$\nu^*_a = \frac{C^*_{12}}{C^*_{22}+C^*_{23}}. \tag{2.7}$$

Here k^* is the transverse bulk modulus for isotropic deformation in the transverse x_2, x_3 plane; G^*_t is the transverse shear modulus for shear in the same plane; G^*_a is the axial shear modulus for shear in longitudinal planes; E^*_a is the axial Young's modulus in fiber direction and ν^*_a is the axial Poisson's ratio associated with uniaxial stress in fiber direction. Other derived moduli are given in [3].

Rigorous expressions for effective elastic moduli on the basis of the composite cylinder assemblage model have been derived in [3]. These expressions, given here in different form, are,

$$k^* = k_1\left[1+\frac{v_2}{\dfrac{k_1}{k_2-k_1}+\dfrac{k_1 v_1}{k_1+\mu_1}}\right] = \frac{k_1(k_2+\mu_1)v_1+k_2(k_1+\mu_1)v_2}{(k_2+\mu_1)v_1+(k_1+\mu_1)v_2} \tag{2.8}$$

$$G^*_a = \mu_1\left[1+\frac{v_2}{\dfrac{\mu_1}{\mu_2-\mu_1}+\dfrac{v_1}{2}}\right] = \mu_1\frac{\mu_1 v_1+\mu_2(1+v_2)}{\mu_1(1+v_2)+\mu_2 v_1} \tag{2.9}$$

$$E^*_a = E_1 v_1 + E_2 v_2 + \frac{4v_1 v_2(\nu_2-\nu_1)^2}{v_1/k_2+v_2/k_1+1/\mu_1} \tag{2.10}$$

$$\nu^*_a = \nu_1 v_1 + \nu_2 v_2 + \frac{v_1 v_2(\nu_2-\nu_1)\left(\dfrac{1}{k_1}-\dfrac{1}{k_2}\right)}{v_1/k_2+v_2/k_1+1/\mu_1}. \tag{2.11}$$

Here it has been assumed that the fibers and matrix are isotropic elastic†. The subscript 2 indicates fibers and the subscript 1 indicates matrix, v is volume fraction and k is the plane

† In the case of phases which are themselves transversely isotropic about an axis in fiber direction, (2.8–12) remain valid with following interpretation: k is transverse bulk modulus, μ is axial shear modulus in (2.9) and transverse shear modulus in all others, E is axial Young's modulus, ν is axial Poisson's ratio in (2.10–11) and is to be interpreted as $(1/2)(1-\mu_t/k)$ in (2.12) where μ_t and k are transverse shear and bulk modulus, respectively. Consequently, all subsequent results for effective complex moduli remain similarly valid for transversely isotropic phases.

strain bulk modulus defined by

$$k = \lambda + \mu$$

where λ is the Lamé modulus and μ the shear modulus.

The results for E_a^* and ν_a^* were given in [2], for hollow fibers, in very complex forms. The much simpler expressions (2.10–11), for solid fibers, were given later by Hill [4].

The situation with respect to the remaining effective modulus G_t^* is much more complicated. Originally, the analysis given in [3] only permitted the establishment of lower and upper bounds for this modulus. Recent work by Hashin and Rosen (to be published) indicates, however, that the upper bound for G_t^* which has been given in [3] may actually be the expression for G_t^* of the composite cylinder assemblage model, for the case of fibers which are stiffer than the matrix.

The G_t^* upper bound was given in complicated implicit form in [3]. It has now been shown that the upper bound, which may be the expression for G_t^*, can be expressed in the much simplified form

$$\begin{aligned}
G_t^* &= G_1 \frac{(1+\alpha v_2^3)(\rho+\beta_1 v_2)-3v_2 v_1^2 \beta_1^2}{(1+\alpha v_2^3)(\rho-\beta_2)-3v_2 v_1^2 \beta_1^2} && \text{(a)}\\
\alpha &= \frac{\beta_1-\gamma\beta_2}{1+\beta_2} \qquad \rho = \frac{\gamma+\beta_1}{\gamma-1} && \text{(b)}\\
\gamma &= \frac{G_2}{G_1} && \text{(c)}\\
\beta_1 &= \frac{1}{3-4\nu_1} \qquad \beta_2 = \frac{1}{3-4\nu_2}. && \text{(d)}
\end{aligned} \tag{2.12}$$

This completes the set of expressions of the five effective moduli (2.3–7). Other important moduli can be expressed in terms of these. For example, the transverse Young's modulus is given by, [3]

$$\begin{aligned}
E_t^* &= \frac{4k^* G_t^*}{k^* + mG_t^*} && \text{(a)}\\
m &= 1 + \frac{4k^* \nu_a^{*2}}{E_a^*}. && \text{(b)}
\end{aligned} \tag{2.13}$$

Numerical results for effective elastic moduli of FRM in which the fibers are circular identical and are arranged in hexagonal or square periodic arrays have been given in the literature [5, 6]. However, for the present purpose of derivation of complex moduli numerical elastic results are useless.

3. COMPLEX MODULI

According to the theory developed in [1] the complex moduli of the FRM are simply found by replacement of phase elastic moduli in (2.8–12) by phase complex moduli. Generally the fibers are elastic and the matrix is viscoelastic. Accordingly the fiber moduli, with subscript 2, are left unchanged and only the matrix moduli, with subscript 1, are replaced.

In order to keep the results as simple as possible it will be assumed, as in [1], that the matrix is elastic in dilatation and is viscoelastic in shear. It will also be assumed that the square of the shear loss tangent of the matrix can be neglected relative to 1, i.e.

$$\tan^2 \delta_\mu \ll 1. \tag{3.1}$$

See [1] for justification of this assumption. It is to be noted that all of the preceding assumptions are only made for convenience. There is no difficulty to proceed without them.

The matrix complex shear modulus is written

$$\tilde{\mu}_1(i\omega) = \mu_1^R(\omega) + i\mu_1^I(\omega) = \mu_1^R(\omega)[1 + i \tan \delta_\mu(\omega)]. \tag{3.2}$$

The complex plane strain bulk modulus may be written as

$$\tilde{k}_1(i\omega) = K_1 + \tfrac{1}{3}\tilde{\mu}_1(i\omega) \tag{3.3}$$

where K_1 is the three dimensional elastic bulk modulus. The form (3.3) is based on the hypothesis of elastic matrix behavior in three dimensional dilatation.

It follows from (3.3) that

$$k_1^R = K_1 + \tfrac{1}{3}\mu_1^R \tag{3.4}$$

$$k_1^I = \tfrac{1}{3}\mu_1^I. \tag{3.5}$$

Now (3.2) and (3.3) are substituted into (2.8) instead of μ_1 and k_1, respectively. Using (3.1–2) and (3.4–5), an easy calculation yields

$$k^{*R} = k_1^R\left[1 + \frac{v_2}{\dfrac{k_1^R}{k_2 - k_1^R} + \dfrac{k_1^R v_1}{k_1^R + \mu_1^R}}\right] \tag{3.6}$$

$$k^{*I} = \frac{1}{3}\mu_1^I\left[1 - \frac{(k_1^R + \mu_1^R)^2 - 4v_1(k_2 - k_1^R)^2}{[k_1^R + \mu_1^R + v_1(k_2 - k_1^R)]^2} v_2\right] \tag{3.7}$$

where

$$\tilde{k}^*(i\omega) = k^{*R} + ik^{*I} \tag{3.8}$$

It is seen that (3.6) is the elastic k^* in terms of real parts of complex moduli. This result and (3.7) can also be derived on the basis of (3.34–35) of [1] which are also applicable for $k^*(i\omega)$.

Expressions (3.6–7) can be greatly simplified if it is assumed that the fibers are rigid. However, this is a problematic assumption for real materials. Generally the fibers are much stiffer than the matrix, but k_1 increases with Poisson's ratio and becomes theoretically infinite for an incompressible matrix. The usual epoxy matrices have a Poisson's ratio of 0·35–0·40 while Glass and Boron fibers have a Poisson's ratio of about 0·2. For Glass–Epoxy FRM where $E_2/E_1 \cong 20$–25 the rigid fiber approximation leads to serious errors for k^*. The situation is better for Boron–Epoxy FRM where $E_2/E_1 \cong 50$–60. It is safer in any event to use (3.6–7) and not the rigid fiber idealization.

Similar procedures are now used to find $\tilde{G}_a^*(i\omega)$ by use of (2.9), (3.1–2). The results are

$$G_a^{*R} = \mu_1^R \frac{\gamma_R(1+c) + 1 - c}{\gamma_R(1-c) + 1 + c} \tag{3.9}$$

$$G_a^{*I} = \mu_1^I \frac{(1-c)[(\gamma_R + 1)^2 + c(\gamma_R - 1)^2]}{[\gamma_R(1-c) + 1 + c]^2} \tag{3.10}$$

where

$$\tilde{G}_a^*(i\omega) = G_a^{*R} + iG_a^{*I} \tag{3.11}$$

$$\gamma_R = \frac{\mu_2}{\mu_1^R} \tag{3.12}$$

$$c = v_2. \tag{3.13}$$

Again (3.9–10) also follow from [1], (3.34).

If the fibers are rigid $\gamma_R \to \infty$ and (3.9–10) become

$$G_a^{*R} = \mu_1^R \frac{1+c}{1-c} \tag{3.14}$$

$$G_a^{*I} = \mu_1^I \frac{1+c}{1-c} \tag{3.15}$$

and consequently

$$\frac{G_a^{*I}}{G_a^{*R}} = \tan\delta^* = \frac{\mu_1^I}{\mu_1^R} = \tan\delta_\mu. \tag{3.16}$$

These results can be derived in a simpler way by first specializing (2.9) to rigid fibers, whence it assumes the form

$$G_a^* = \mu_1 \frac{1+c}{1-c} \tag{3.17}$$

and then using the correspondence principle. This procedure directly leads to (3.14–16) without the use of (3.1).

These results are in accordance with the general results obtained in [1] (equations 3.20–21) for the complex shear modulus of isotropic composites, which consist of a viscoelastic incompressible phase and a rigid phase. Similar general arguments can be given for the present case. It should first be noted that the axial elastic shear modulus ${}^eG_a^*$ of a unidirectionally FRM depends only upon the shear moduli of the constituents and not on their Poisson's ratios. This has been implicitly shown in [7]. Consequently for rigid fibers

$${}^eG_a^* = {}^eG_a^*({}^e\mu_1, \text{phase geometry}) \tag{3.18}$$

which may be rewritten in the form

$${}^eG_a^* = {}^e\mu_1\psi_a \tag{3.19}$$

where ψ_a is a nondimensional function of the phase geometry only. Applying the correspondence principle to (3.19) it follows that

$$G_a^{*R} = \mu_1^R\psi_a, \qquad G_a^{*I} = \mu_1^I\psi_a \tag{3.20}$$

$$\tan\delta^* = \tan\delta_\mu \tag{3.21}$$

where the loss tangents in (3.21) are defined in (3.16).

Note that in the present case *it is not necessary to assume that the matrix is incompressible.*

The results (3.14–16) are merely a special case of (3.20–21) for the composite cylinder assemblage model.

In order to assess the range of approximate validity of (3.24) in the case when the fibers are not rigid, yet much stiffer than the matrix, we compute the difference between the effective shear loss tangent based on (3.9–10), and the matrix loss tangent. The result is

$$\frac{G_a^{*I}}{G_a^{*R}} - \tan\delta_\mu = \frac{4\gamma_R c}{[\gamma_R(1+c)+1-c][\gamma_R(1-c)+1+c]}\tan\delta_\mu \tag{3.22}$$

It is seen that the difference is of order γ_R^{-1} and thus becomes quite small for elevated γ_R, i.e. stiff fibers. Thus for stiff fibers the FRM loss tangent is very nearly that of the matrix while, however, the real and imaginary parts of the effective complex modulus are not computed on the basis of the rigid fiber idealization but on the basis of the elastic fiber expressions (3.9–10). It will be recalled that a similar phenomenon has been observed in [1] for the complex shear modulus of composites consisting of stiff particles (sand) and soft matrix (epoxy).

We now consider transformation of the results (2.10–11) to dynamic viscoelasticity. There is of course no intrinsic difficulty to directly use the correspondence principle as has been done before. However, the relative complexity of (2.10–11) results in very tedious calculations for separation of the effective complex moduli into real and imaginary parts. It is fortunately possible to introduce simplifications. It is known that for the usual elastic FRM the third term on the right side of (2.10) is numerically insignificant in comparison to the first two terms. Some numerical results in this respect have been quoted in [2]. Consequently this term may be safely neglected. Then is left the simple expression

$$E_a^* = E_1 v_1 + E_2 v_2 \tag{3.23}$$

Indeed (3.23) is a rigorous result for *any fiber geometry* when the matrix and fiber Poisson's ratios are the same. This has first been stated in [3]. Experimental results are in excellent agreement with (3.23).

Application of the correspondence principle to (3.23) leads to the simple results

$$\tilde{E}^*(i\omega) = \tilde{E}_1(i\omega)v_1 + E_2 v_2 \tag{3.24}$$

$$E^{*R} = E_1^R v_1 + E_2 v_2 \tag{3.25}$$

$$E^{*I} = E_1^I v_1 \tag{3.26}$$

$$\tan\delta_E^* = \frac{E^{*I}}{E^{*R}} = \frac{\tan\delta_E}{1+\dfrac{E_2}{E_1^R}\dfrac{v_2}{v_1}} \tag{3.27}$$

where

$$\tan\delta_E = \frac{E_1^I}{E_1^R} \tag{3.28}$$

It is seen that for the usual stiff fibers the denominator in (3.27) becomes large and therefore $\tan\delta_E^*$ is much smaller than $\tan\delta_E$. This indicates that the viscoelastic effect in axial stressing is insignificant, as has also been found in the static case, [2].

It is to be expected that (3.24–28) should apply with high accuracy for any fiber geometry.

The situation with respect to (2.11) is not quite as simple. Still even here the third term in the right side of (2.11) may be neglected, although with less accuracy than in (2.10). This results in

$$\nu_a^* \cong \nu_1 v_1 + \nu_2 v_2 \tag{3.29}$$

$$\tilde{\nu}_a^*(i\omega) \cong \tilde{\nu}_1(i\omega) v_1 + \nu_2 v_2 . \tag{3.30}$$

The expression $\tilde{\nu}_1(i\omega)$ is to be interpreted as

$$\tilde{\nu}_1(i\omega) = \frac{3K_1 - 2\mu_1(i\omega)}{2[3K_1 + \mu_1(i\omega)]} \tag{3.31}$$

where it has been assumed that the matrix is elastic in dilatation. Using (3.2) and (3.1) in (3.31) there results

$$\nu_1^R = \frac{3K_1 - 2\mu_1^R}{2(3K_1 + \mu_1^R)} \tag{3.32}$$

$$\nu_1^I = \frac{9\mu_1^I K_1}{2(3K_1 + 2\mu_1^R)^2} \tag{3.33}$$

$$\tilde{\nu}_1(i\omega) = \nu_1^R + i\nu_1^I . \tag{3.34}$$

Substitution of (3.32–34) into (3.30) gives the real and imaginary parts of $\tilde{\nu}_a^*(i\omega)$.

It is interesting to note that for incompressible matrix (3.30) reduces to the real result

$$\tilde{\nu}_a^*(i\omega) = \tfrac{1}{2} v_1 + \nu_2 v_2 \tag{3.35}$$

which is frequency independent and thus provides the constant real part while the imaginary part vanishes.

Finally, we consider the complex effective transverse shear modulus $\tilde{G}_t^*$. In principle, there is of course no difficulty to exploit (2.12) on the basis of the correspondence principle to find $\tilde{G}_t^*(i\omega)$, but this requires cumbersome algebra and results in complicated expressions. If we adopt again the assumption (3.1) and also assume that the matrix is viscoelastic in shear only then it follows from [1], equation (3.34) that the real part G_t^{*R} is simply given by replacement of μ_1 by μ_1^R, and consequently replacement of ν_1 by (3.32), in (2.12). The loss tangent can in principle be computed by use of [1], (3.35) and of (2.12) but the required calculations are very heavy.

Considerable simplification is obtained if it is assumed that the fibers are perfectly rigid. In this event $\gamma \to \infty$ in (2.12). Simplifying (2.12) accordingly and substituting real parts of matrix complex moduli for matrix elastic moduli we obtain

$$\begin{aligned} G_t^{*R} &= G_1^R \frac{(1-v_2^3)(1+\beta_1 v_2) - 3v_2 v_1^2 \beta_1^2}{(1-v_2^3)(1-v_2) - 3v_2 v_1^2 \beta_1^2} \\ \beta_1 &= \frac{1}{3-4\nu_1^R} . \end{aligned} \tag{3.36}$$

Applying [1], equation (3.35), to (3.36) it is found that

$$\tan \delta^*_{G_t} = \left[1 - \frac{4}{3}(1+v_1^R)(1-2v_1^R)\beta_1^2 v_1^3 v_2 \times \frac{1-v_2^3+3v_2v_1\beta_1(\beta_1+2)}{(1-v_2^3-3v_2v_1\beta_1^2)[(1-v_2^3)(1+\beta_1 v_2)-3v_2v_1^2\beta_1^2]} \right] \tan \delta_\mu \tag{3.37}$$

Numerical calculations show that the parenthesis of (3.37) is generally only a few per cent smaller than unity. Therefore

$$\tan \delta^*_{G_t} \sim \tan \delta_\mu \tag{3.38}$$

Such an approximation is also to be expected for fibers which are not entirely rigid, but very much stiffer than the matrix.

For incompressible matrix and rigid fibers it follows again by previous arguments that the effective elastic shear modulus is representable in the form

$$^eG_t^* = {}^e\mu_1 \psi_t \,(\text{phase geometry}). \tag{3.39}$$

Then the complex shear modulus is given by

$$\begin{aligned} &\tilde{G}^*(i\omega) = \tilde{\mu}_1(i\omega)\psi_t && \text{(a)} \\ &G^{*R} = \mu_1^R \psi_t, \qquad G^{*I} = \mu_1^I \psi_t && \text{(b)} \end{aligned} \tag{3.40}$$

and the transverse shear loss angle is that of the matrix

$$\tan \delta^* = \tan \delta_\mu \tag{3.41}$$

The result (3.41) is similar to (3.38). Indeed it is seen that for incompressible matrix $v_1^R = \frac{1}{2}$ and (3.37) reduces to (3.41).

The results (3.38) and (3.41) permit computation of G_t^{*I}, approximately, for FRM with stiff fibers. To do this G_t^{*R} is first computed by use of (2.12), as has been explained above. Then

$$G_t^{*I} \sim G_t^{*R} \tan \delta_\mu . \tag{3.42}$$

To obtain results for the complex effective transverse Young's modulus $\tilde{E}_T^*(i\omega)$ we note that the complex counterpart of (2.13) is

$$\begin{aligned} &\tilde{E}_T^*(i\omega) = \frac{4\tilde{k}^*(i\omega)\tilde{G}_t^*(i\omega)}{\tilde{k}^*(i\omega)+m(i\omega)\tilde{G}_t^*(i\omega)} && \text{(a)} \\ &m(i\omega) = 1 + \frac{4\tilde{k}^*(i\omega)[v_a^*(i\omega)]^2}{\tilde{E}_a^*(i\omega)} && \text{(b)} \end{aligned} \tag{3.43}$$

which permits computation of real and imaginary parts in terms of previously discussed complex moduli.

4. STRUCTURAL APPLICATION

As has been discussed in [1] the present theory assumes that within a sufficiently narrow frequency range dynamical behavior of heterogeneous bodies may be macroscopically approximated by classical continuum mechanics with effective moduli replacing homogeneous moduli. This has been termed the first approximation of dynamics of composites.

As is well known (see e.g. [8]) the theory of steady state vibrations of viscoelastic structures can be reduced to elastic vibration theory with complex moduli replacing the elastic moduli. According to the present hypothesis the same can be done for fiber reinforced viscoelastic structures.

As a simple example consider forced torsional vibrations of a fiber reinforced cylinder of length l. Let $\theta(x, t)$ be the angle of twist. The cylinder is built in at one end thus

$$\theta(0, t) = 0 \tag{4.1}$$

At its other extremity it is subjected to the forcing torque

$$T(l, t) = T_0 \, e^{i\omega t} \tag{4.2}$$

where ω is the frequency and T_0 the amplitude. For an elastic cylinder with axial shear modulus G_a^* the angle of twist at $x = l$ is then given by

$$\theta(l, t) = \frac{\alpha T_0}{\omega G_a^*} \tan\left(\frac{\omega l}{\alpha}\right) e^{i\omega t} \tag{4.3}$$

where

$$\alpha^2 = \frac{G_a^* C}{\rho I}. \tag{4.4}$$

In (4.4) $G_a^* C$ is the torsional rigidity, ρ the density and I the polar moment of inertia.

For a viscoelastic fiber reinforced cylinder, G_a^* must be replaced by the complex modulus $\tilde{G}_a^*(i\omega)$. For simplicity it is assumed that the fibers are rigid and thus the results (3.14–15) apply.

A straightforward analysis yields the following results: If the forcing torque is sinusoidal i.e.

$$T(l, t) = T_0 \sin \omega t \tag{4.5}$$

then the angle of twist is given by

$$\theta(l, t) = \frac{\tilde{\alpha} T_0}{\omega G_a^{*R} C} \frac{\sin^2(2\beta) + \sinh^2(2\gamma)}{\cos(2\beta) + \cosh 2\gamma} \sin(\omega t - \phi) \tag{4.6}$$

where

$$\phi = \tan^{-1}\left[\frac{\sinh(2\gamma)}{\sin(2\beta)}\right] + \delta/2 \tag{4.7}$$

$$\tilde{\alpha}^2 = \frac{G_a^{*R} C}{\rho I} \tag{4.8}$$

$$\beta = \frac{\omega l}{\tilde{\alpha}} \cos \delta/2 \tag{4.9}$$

$$\gamma = \frac{\omega l}{\tilde{\alpha}} \sin \delta/2. \tag{4.10}$$

Here ρ is the average† density, G_a^{*R} is given by (3.14) and δ is the matrix shear loss angle. For details of the derivation see [9].

A numerical plot of the amplitude Amp$[\theta(l, t)]$ vs. frequency ω is shown in Fig. 1. The data used are as follows:

$l = 5{\cdot}0$ ft.

$d = 4{\cdot}0$ in.—diameter of circular section.

$\rho = 3{\cdot}0$—average density relative to water.

$\mu_1^R(0) = 0{\cdot}5 \times 10^6$ psi—elastic matrix shear modulus.

Tan $\delta = 0{\cdot}1$—matrix loss angle, assumed frequency independent.

$\mu_1^R(\omega) = \mu_1^R(0)[1+\frac{1}{4}\log_{10}\omega]$; $1 < \omega < 10^4\ \text{sec}^{-1}$—variation of real part of matrix complex shear modulus with frequency.

$c = 0{\cdot}6$—volume fraction of fibers.

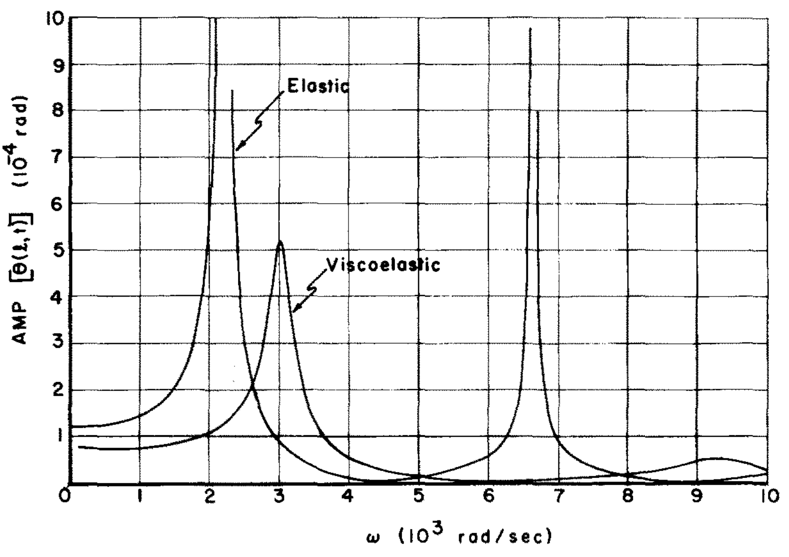

FIG. 1. Amplitude of angle of twist.

Also plotted in Fig. 1 is amplitude of elastic twist angle vs. frequency, assuming initial real part of complex modulus, $\mu_1^R(0)$, as matrix shear modulus for the whole frequency range.

It is seen that resonance peaks are very quickly smoothened out by the viscoelastic effect.

Further structural applications are given in [9]. It has there been shown that in bending vibrations the viscoelastic effect in smoothening out resonance peaks is much less pronounced than in torsional vibrations.

† Recent unpublished work by the writer indicates that it is necessary to use an effective density which is not the average density.

5. CONCLUSION

Viscoelastic dynamic behavior of fiber reinforced materials has been analyzed on the basis of a first approximation. This approximation involves the assumption that classical dynamic theory applies to composites with replacement of homogeneous physical constants by effective constants of the composite.

Such a theory can be expected to give a reasonable approximation for dynamic behavior of composites only for restricted frequency ranges. To take into account the dispersive nature of the composite new theories are needed. For such investigations in the elastic case see the papers of Herrmann and Achenbach [10, 11]. It is to be expected that any nonclassical theory for dynamic behavior of elastic composites can be carried over into a corresponding viscoelastic nonclassical theory by a correspondence principle similar to the one used here.

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Абстракт—Используется соответствующий принцип, выведенный в предыдущей работе, для определения выражений эффективных комплексных модулей, для материалов упрочненных волокнами, на основе выражений для эффективных упругих модулей материалов также упрочненных волокнами.